A SIMPLIFIED ORTHOTROPIC FORMULATION OF THE VISCOPLASTICITY THEORY BASED ON OVERSTRESS

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An orthotropic, small strain viscoplasticity theory based on overstress is presented. In each preferred direction the stress is composed of time (rate)-independent (or plastic) and viscous (or rate-dependent) contributions. Tension-compression asymmetry can depend on direction and is included in the model. Upon a proper choice of a material constant one preferred direction can exhibit linear elastic response while the other two deform in a viscoplastic manner.

INTRODUCTION

Recently directionally solidified alloys, nickel base single crystal superalloys and other anisotropic metallic composites have attracted interest for use in gas turbines and other high temperature applications. The usual high temperature phenomena such as creep, relaxation, rate sensitivity, recovery and aging found in nearly isotropic materials are also present in these materials. However, all these properties are now dependent on direction.

For the prediction of life of components made of anisotropic materials and operating at elevated temperature the deformation behavior must be known in addition to anisotropic damage accumulation laws. It is the purpose of this paper to introduce an orthotropic version of the viscoplasticity theory based on overstress (VBO), (the transversely isotropic case can be recovered as a specialization). The uniaxial and the isotropic version of VBO were introduced previously [1,2]. The theory is of the unified type (plasticity and creep are not represented by separate constitutive equations) and does not employ the concepts of a yield surface and associated loading and unloading conditions. In the present form of the theory aging and recovery are not accounted for but can be added if need arises.

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The orthotropic formulation was derived with the help of tensor function representation theorems [3] and the uniaxial version of the VBO. Simplicity was a goal as long as it was consistent with the necessity of modeling key material phenomena. The tensor function approach is not restricted to the orthotropic case and can be applied to other material symmetries as well.

UNIAXIAL PROPERTIES OF VBO INCLUDING ASYMPTOTIC SOLUTIONS

In the formulation of VBO special consideration was given to the modeling of elastic regions in addition to the usual time-dependent properties [1]. A useful property of the system of nonlinear differential equations is the existence of asymptotic solutions which are algebraic expressions. They apply mathematically at infinite time in a constant strain rate or creep, or relaxation test. However, it is our experience that these asymptotic solutions can be used with confidence when plastic flow is fully developed in a tensile test [1,4].

A schematic of the properties of the model in a tensile test is given in Fig.1. The evolution of the stress σ, the equilibrium stress g (which is reached when all rates approach zero) and of the quantity $f = E_t\epsilon$ are shown. It is introduced for modeling a nonzero slope E_t in the plastic region even when the asymptotic solutions are attained. The asymptotic values are indicated by { } in Fig.1. It is seen that the stress consists of $\{\sigma - g\}$, the time-independent or viscous contribution, of $\{g - f\}$ which represents the time-independent or plastic part and of the portion which grows linearly with ϵ; it is termed the hardening contribution. It is zero when the tangent modulus E_t is set to zero, see [1,2] for further details. In a neighborhood of the origin, σ and g almost coincide and nearly elastic behavior is represented.

In the formulation of the anisotropic version of VBO the elastic properties can depend on direction. In addition, it was felt necessary to have separate directional properties for the viscous, the plastic and the hardening contributions to the stress. (It is important to note that the theory does not separately formulate plastic and time-dependent constitutive equations. However, the asymptotic solutions of the theory permit such a distinction.) The reason for this distinction lies in the realization that different material constituents may be used in different directions (example; directionally solidified alloys) or that the microstructure may develop an orientation dependence. Moreover, fibers with predominantly elastic behavior may run in one direction and the viscoplastic matrix may control the behavior in other directions.

AN ORTHOTROPIC VISCOPLASTICITY THEORY BASED ON OVERSTRESS

In [6] a fully invariant theory is developed with arbitrary orientation of the principal material axes relative to the coordinate

system used in the representation of the tensors. Presently we assume that the coordinate system in which the tensor components are given coincides with the material axes. Vector notation is used with

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3, \quad \sigma_{23} = \sigma_4, \quad \sigma_{13} = \sigma_5 \quad \text{and} \quad \sigma_{12} = \sigma_6 \qquad (1)$$

and with a similar convention for the small strain ϵ except that engineering shear strains are used for the vector components ϵ_4 through ϵ_6.

The evolution of the stress is governed by

$$d\sigma/dt = C\{d\epsilon/dt - d\epsilon^{in}/dt\} \qquad (2)$$

where C is the matrix of elastic constants. The inverse of C, C^{-1} is given by, see [5]

$$
C^{-1} = \begin{bmatrix}
1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\
 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\
 & & 1/E_3 & 0 & 0 & 0 \\
\text{Symmetric} & & & 1/G_{23} & 0 & 0 \\
 & & & & 1/G_{13} & 0 \\
 & & & & & 1/G_{12}
\end{bmatrix} \qquad (3)
$$

The inelastic strain rate $d\epsilon^{in}/dt$ is represented by

$$d\epsilon^{in}/dt = K[\Gamma] \, Rx \qquad (4)$$

where the positive function $K[\Gamma]$ is a repository of viscous effects ($K[\Gamma] = 1/(Ek[\Gamma])$) where $k[\Gamma]$ is the viscosity function used in [1]). The dimensionless components of the matrix R are called the inelastic lateral ratios. The invariant Γ is defined as

$$\Gamma = (x^T V V x)^{1/2} + a^T V x \qquad (5)$$

with $a^T = [a_1 \ a_2 \ a_3 \ 0 \ 0 \ 0]$, where the dimensionless components a_1 are zero when the viscous effects are the same in tension and compression.

$\underset{\sim}{V}$ is the matrix of the viscous lateral ratios. The overstress is $\underset{\sim}{x} = \underset{\sim}{\sigma} - \underset{\sim}{g}$. The equilibrium stress evolves according to

$$d\underset{\sim}{g}/dt = \psi[\Gamma]\underset{\sim}{B}[\Gamma](d\underset{\sim}{\varepsilon}/dt - \theta^2 d\underset{\sim}{\varepsilon}^{in}/dt) \qquad (6)$$

This growth law is very similar to the one used in [1,2]. The invariant θ is given by

$$\theta = \left((\underset{\sim}{g}^T\underset{\sim}{S}\,\underset{\sim}{S}\,\underset{\sim}{g})^{1/2} + \underset{\sim}{b}^T\underset{\sim}{S}\,\underset{\sim}{g}\right)/A \qquad (7)$$

with $\underset{\sim}{b}^T = [b_1\ b_2\ b_3\ 0\ 0\ 0]$. The dimensionless components b_i are zero when the plastic effects in tension and compression are equal. The analysis of the asymptotic behavior of the uniaxial equivalent of (6) in [1] shows that $\{g - f\}$ in Fig.1 equals the constant A which has the dimension of stress. The dimensionless components of the matrix $\underset{\sim}{S}$ are the plastic lateral ratios. The dimensionless components of $\underset{\sim}{B}$ are called shape ratios and are initially equal to the components of $E_1\underset{\sim}{C}^{-1}$, called the elastic ratios. The positive, decreasing shape function $\psi[\Gamma]$ has the dimension of stress with $\psi[0]$ slightly less than the elastic modulus E_1, see [1]. For simplicity the tangent modulus E_t was set equal to zero so that f in Fig.1 is zero and all the stress-strain curves become ultimately horizontal.

Due to orthotropy, the matrices $\underset{\sim}{R}$, $\underset{\sim}{V}$, $\underset{\sim}{S}$ and $\underset{\sim}{B}$ all have the same representation as the matrix $\underset{\sim}{C}^{-1}$ in (3) and have therefore nine independent components. The components of each matrix can be selected independently to model the observed directional dependence of the various material properties.

The initial elastic properties are controlled by $\underset{\sim}{C}$ as in the case of elasticity. The evolution of the inelastic strain rates are influenced by $\underset{\sim}{R}$ and $\underset{\sim}{V}$. They also contribute to the asymptotic overstress $\{\underset{\sim}{x}\}$, see Fig.1, given by

$$\{\underset{\sim}{x}\} = \underset{\sim}{R}^{-1}\ d\underset{\sim}{\varepsilon}/dt/K[\{\Gamma\}] \qquad (8)$$

The asymptotic time-independent or plastic contribution to the stress is controlled by the invariant θ through

$$\{\theta\}^2 = 1 \qquad (9)$$

and it is seen from (7) that the directional properties are controlled by $\underset{\sim}{S}$ alone.

Detailed analysis in [6] shows that the matrix $\underset{\sim}{B}$ together with the shape function $\psi[\Gamma]$ controls the "knee" of the stress-strain curves in different directions.

A simplified version which has been shown to be useful [6] is to set $\underset{\sim}{R} = \underset{\sim}{B}^{-1} = E_1 \underset{\sim}{C}^{-1}$ and to choose $\underset{\sim}{S}$ and $\underset{\sim}{V}$ independently. This choice permits the independent adjustment of the viscous and plastic asymptotic contributions to the stress. Within this choice it is possible to model

      i)   purely elastic behavior under a hydrostatic state of stress,

     ii)   linear elastic behavior in any of the preferred directions while the other directions behave in a viscoplastic manner.

This last property is very useful for modeling fiber reinforced materials. It should also be stressed that the theory permits the modeling of tension/compression asymmetry which depends on direction through the dimensionless vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

The capabilities of the theory are demonstrated in Figs.2 through 4. They depict the response of a transversely isotropic material to a constant strain rate tensile test in the 1- and 3-directions, respectively. In Fig.2 $\underset{\sim}{R} = \underset{\sim}{S} = \underset{\sim}{V} = \underset{\sim}{B}^{-1} = E_1\underset{\sim}{C}^{-1}$ and the evolution of the stress and of the equilibrium stress are governed by the values of $\underset{\sim}{C}^{-1}$. It is seen that the elastic modulus, the stress and the overstress in the 3-direction are always larger than in the 1-direction. When S_{33} and V_{33} are set equal to zero (all other quantities are the same as in Fig.2) the response in the 3-direction is nearly linear elastic whereas that in the 1-direction is unaffected, see Fig.3. When S_{33} is set equal to 0.5 (instead of 50/35 used in Fig.2; all other quantities are unchanged from Fig.2) the curves of Fig.4 result. This choice will increase $\{\varepsilon_{33}\}$, the plastic or time independent part of the stress, but will leave the overstress, the viscous contribution to the stress, unchanged. Due to the nature of the constants the equilibrium solution has not been attained within the limits of the graph in Fig.4.

The above represents only part of the capabilities of the theory developed in [6]. It includes an incompressible inelastic, deviatoric formulation. Further developments are given in [7]. The theory needs to be applied to real anisotropic materials so that the material functions and constants can be identified and the usefulness of the theory be demonstrated.

ACKNOWLEDGEMENTS

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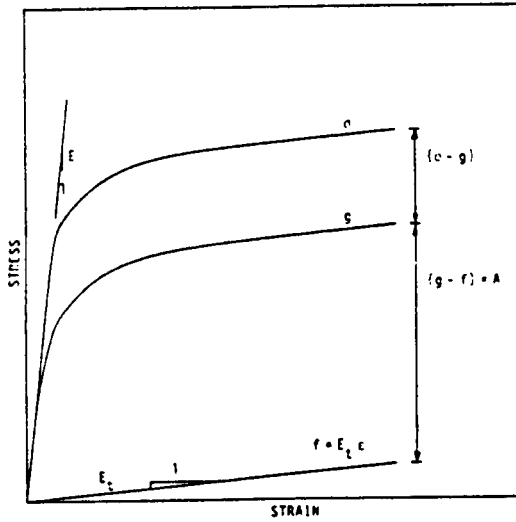
Figure 1. Schematic showing the viscous $(\sigma-g)$, plastic $(g-f)$ asymptotic contributions to the stress. The hardening contribution $f = E_t\epsilon$ is also shown. In this paper $E_t = 0$.
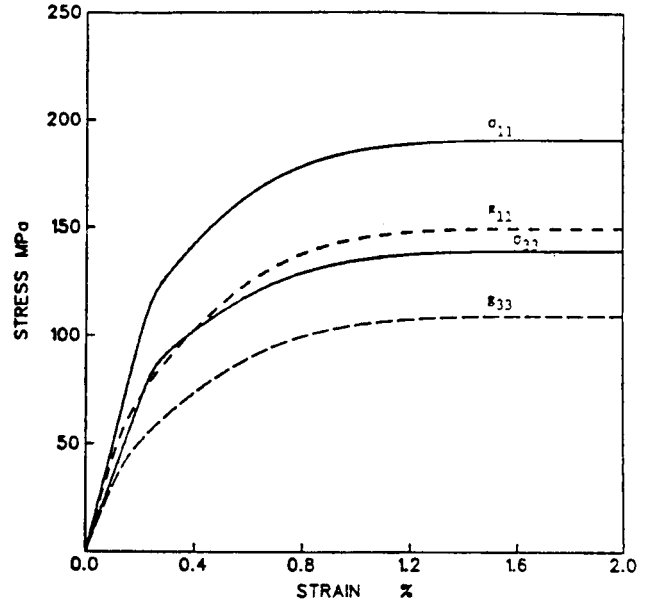

Figure 2. Uniaxial tensile tests in the 1- and 3-directions, respectively. Strain rate is 10^{-4} s^{-1}. $\underline{R} = \underline{S} = \underline{V} = \underline{B}^{-1} = E_1\underline{C}^{-1}$.
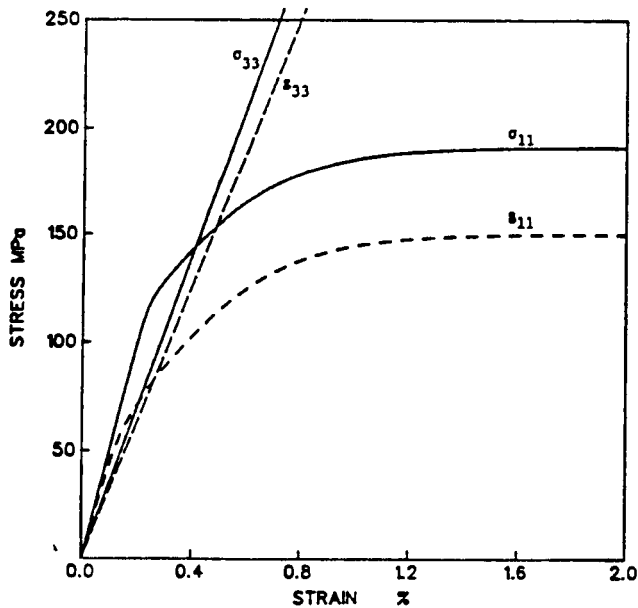

Figure 3. Same as Fig. 2 except that $S_{33} = V_{33} = 0$. Nearly linear elastic response in the 3-direction results. The response in the 1-direction is unaltered.
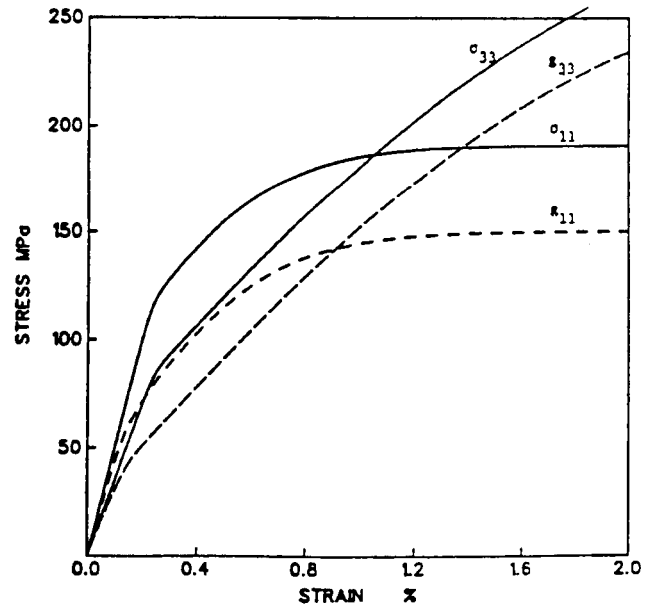

Figure 4. Same as Fig. 2 except that $S_{33} = 0.5$ instead of 50/35 used in Fig. 2. As a consequence (g_{33}) is increased without changing $(\sigma_{33}-g_{33})$. The response in the 1-direction is unaltered.

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